FIRST INTERNATIONAL CONFERENCE
Modelling and Development of Intelligent Systems
Sibiu - Romania, 22-25 October, 2009

Some approximations in the bar deflection analysis, II

Nicolae Boja, Georgeta Brăiloiu, Remus Ene

Abstract

The aim of this paper is to approximate the strain energy of deformation and to determine the work of a charge P acting **oblique** on an elastic slender prismatic bar, by using the method of bar discretization. The discrete distribution of the deformation state is described by means of recurrence formulas. Also, in the case of a curved bar (cross-section is a curvilinear trapezium), a dependence on the principal curvatures of the middle surface of the bar is put in evidence.
The study is made for a bar viewed as a simple uniform and isotropic material body ([12]), assumed as a trivial manifold without boundary endowed with a single coordinate chart.
Finally, a numerical example for a bar with known physical and geometrical characteristics is given.

Keywords: strain energy, intrinsic representation, deflection map and angle of deflection, principal curvatures.

1. Introduction.

Consider an elastic slender bar of length L with a fixed endpoint A and a free endpoint B loaded with a force P ; $\overline{\mathrm{AB}}$ denotes its initial centroidal axis, as a vertical straight line segment.

Usually the physical properties of the bar by the elasticity modulus E and by the inertial moment I are given, and the geometrical ones by dimensions, by the cross-section shape, and by the centroidal axis aspect (especially by the curvature $1/R$) are described. The magnitude of deformation at each point of the bar also depends on the charge value P as well on the direction of the force action, that is of the load vector **P**, ($P = \left\| \mathbf{P} \right\|$).

Especially two directions of the load vector action are of interest for the technicians:

I. The action of **P** is vertically,

II. The action of **P** is oblique, but its direction is passing through a fixed point of the plane of deformation.

Because the first case was treated in our paper [6], in this paper will be approached only the case II.

In the first case the bar is subjected (at time t = 0) to an axial compression which will be constant in time (t > 0). In the second one the compression is not axial.

Many studies about the elastic bar deflections by a long sequence of authors were made (see [1], [2], [3], [4], etc.). They analyzed especially the deflections of prismatic bars with deformed centroidal axis relative to a 'Cartesian coordinate system', associated to a reference configuration, i.e. to an embedding

$$\varphi_0 : \overline{\mathrm{AB}}\ \alpha\ \ \mathbf{E}^2,$$

where $\mathbf{E}^2$ is the Euclidean 2- space, called *plane of deformation*.

Here we have to mention the fact that because a bar is usually viewed as a simple uniform and isotropic material body, it can be assumed as a trivial manifold without boundary for which is enough to consider a single coordinate chart φ_0 that cover the wholly body. So, $\mathbf{E}^2$ can be initially endowed with an orthogonal frame as (Axy), with the origin at fixed end A and with $\overline{\mathrm{AB}}$ along to the axis (Ay).

Different from the former papers in this paper we approach the study of the bar deflections having in view a deformed centroidal axis with respect to an 'intrinsic representation'. Until a point this study can be made in a similar way both for bars with rectangular and with curved (but symmetrical) cross-section, that is independently of the cross-section shape.

This last aspect permit us to correct the differential equation of the deformed centroidal axis that follows from the expression $M = EI \cdot 1/R$ of the bending moment.

Thus, the geometric aspect of the deformation will be described with respect to a new coordinate frame of the Euclidean affine plane $\mathbf{E}^2$. So, during deformation we consider a mobile orthogonal frame (OXY) with the origin O placed (at each time t > 0) at the free end of the centroidal axis of bar and with axes (OX) and (OY) parallel with the initial axes (Ay) and (Ax), respectively, but having opposite orientation.

With respect to this frame the deformed centroidal axis $\overline{\overline{\mathrm{AO}}}$ can be looked as an image of the *deflection map*

$$\phi : \overline{\mathrm{AB}}\ \alpha\ \ \overline{\overline{\mathrm{AO}}},$$

defined by means of an intrinsic equation

$$\psi = \psi(s), \quad (s \in [0, L]), \tag{1}$$

where $s = \lambda(\mathrm{O}\widetilde{\mathrm{M}})$ is the arc length from the origin point $\mathrm{O} = \phi(\mathbf{B})$ to an arbitrary point $\widetilde{\mathrm{M}} = \phi(\mathrm{M})$, $(\mathrm{M} \in \overline{\mathrm{AB}})$, and ψ denotes the angle measure of the tangent straight line at $\widetilde{\mathrm{M}}$ to the centroidal axis of the deformed bar with the positive orientation of the axis (OX), called *angle of deflection* at M. We mention that here ϕ can be seen as a composed map, $\phi = \varphi \circ \varphi_0$, where φ is another configuration in $\mathbf{E}^2$ of the centroidal axis $\overline{\mathrm{AB}}$ which becomes at time t (> 0) a curvilinear arc $\overline{\overline{\mathrm{AO}}}$; (1) is called *intrinsic equation* of the deformed bar.

Further, we use the exact expression of the curvature

$$\frac{1}{R} = \frac{d\psi}{ds} \tag{2}$$

instead of the approximate one d^2y / dx^2, which is frequently used by many researchers in order to obtain a second order differential equation that can be easily integrated. In such a case the general method adopted to reduce he nonlinearity degree of the differential equation describing the bar deformation makes use of the Cebishev's polynomials.

The deformations of a slender bar such as is considered above by an equation of Bessel type are described:

$$\frac{d^2\psi}{ds^2} + \frac{w}{EI} s \cdot \cos(\beta - \psi) = 0, \tag{3}$$

where w is the specific weight per unit length and β is the angle measure of undeformed bar with the horizontal plane.

Exact solutions of (3) are known for the cases of horizontal (β = 0) or vertical ($\beta = \pi / 2$) initial positions of the bar.

For instance, in the case of a vertical bar the deflection ϕ by a linear equation is described (see Denman and Schmidt, [1970]) :

$$\frac{d^2\psi}{ds^2} + \mathbf{k}_1{}^2 s \cdot \psi = 0, \tag{4_1}$$

where

$$\mathbf{k}_1{}^2 = 2wJ_1(\alpha) / \alpha \cdot EI \tag{5}$$

is a constant depending on the Bessel function

$$J_1(\alpha) = \frac{1}{\pi\alpha} \int_{-\alpha}^{\alpha} \psi \sin\psi \cdot (\alpha^2 - \psi^2)^{-1/2} d\psi .$$

We note, this equation is obtained from (3) taking $\beta = \pi/2$ and $\sin\psi = \frac{2J_1(\alpha)}{\alpha}\cdot\psi$ on the symmetric real interval $[-\alpha, \alpha]$.

In this case the known general solution of (4_{I}) is

$$\psi = As^{1/2}J_{1/3}(\frac{2}{3}ks^{3/2}) + Bs^{1/2}J_{-1/3}(\frac{2}{3}ks^{3/2}),$$

where A, B and k are constants.

One can see that this exact solution is not quite a simple one because of more difficulties of the integrals computing. This is the reason we tray to find an alternative solution by means of the bar discretization; the solution so obtained is consisting in a recursive approximation.

Finally, we end the introduction with the following remark.

In this study the bar is considered to be vertical. However, the differential equation (4_{I}) will be used only in the case when the load **P** is vertically too, while when direction of **P** is oblique it is better to make use of another differential equation of the deformed centroidal axis obtained from the bending moment equation. So, expressing the moment M in two different ways, we have

$$EI\cdot\frac{d\psi}{ds} = -P_X\cdot Y + P_Y\cdot X, \tag{4_{II}}$$

where P_X, P_Y are the projections of **P** on the axes (OX), (OY), respectively, and X, Y are the coordinates of the point $\widetilde{\mathrm{M}}$ with respect to these axes.

In this case, another positive constant will be useful in our study:

$$\mathbf{k}_0^2 = P/EI. \tag{6}$$

Let C be the intersection point of the straight line which defines the direction of **P** with the (Ay) axis of the initial frame and denote by c its distance to the free end B of the bar. If the measure of the angle of the previous two straight lines is accepted to be small, we can approximate by P the projection P_X, i.e. $P_X \cong P$. Thus, if δ is the horizontal displacement during the bar deflection of the free end B, i.e. $\delta = d(\mathrm{O},(\mathrm{Ay}))$ (see Fig.1), by a geometrical reason, it follows $P_Y/\delta = P_X/c$, which implies that $P_Y \cong P\cdot\frac{\delta}{c}$.

On the other hand we know that at each point $\widetilde{\mathrm{M}}$ (X, Y) the relations

$$\frac{dX}{ds} = \cos\psi, \quad \frac{dY}{ds} = \sin\psi$$

are specific for a plane curve given in an intrinsic representation, such as (1).

Thus, by derivation of the equation (4_{II}) with respect to s, this one can be

$$\frac{d^2\psi}{ds^2} = -\mathbf{k}_0^2(\sin\psi - \frac{\delta}{c}\cos\psi)$$

or, still,

$$\frac{1}{2}\frac{d}{ds}(\frac{d\psi}{ds})^2 = -\mathbf{k}_0^2(\sin\psi - \frac{\delta}{c}\cos\psi)\frac{d\psi}{ds}$$

Multiplying this equation by ds and integrating it one obtains

$$\frac{1}{2}(\frac{d\psi}{ds})^2 = \mathbf{k}_0^2(\cos\psi + \frac{\delta}{c}\sin\psi) + C\,; \tag{7}$$

the integration constant C can be found with the help of boundary conditions (8), given bellow.

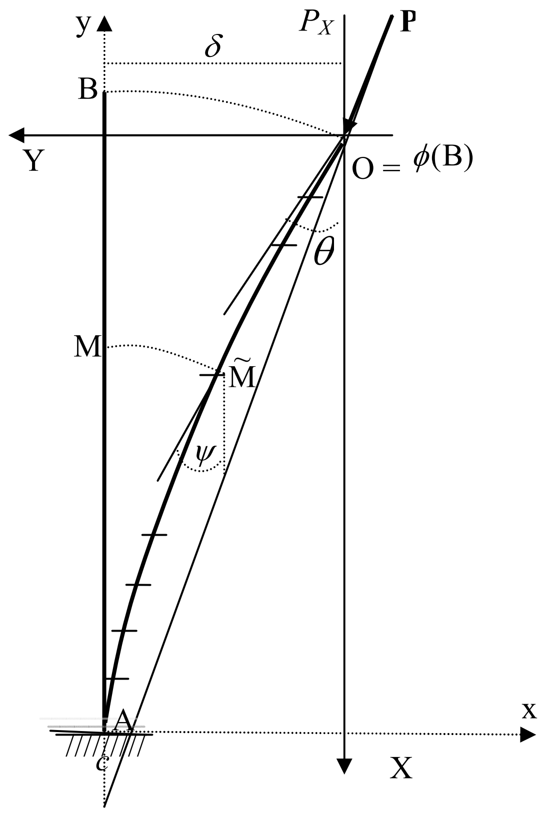

Fig.1. Bar deflection under the action of an oblique load **P**

2. Boundary value problems and the bar discretization.

Consider a deformed bar and assume the deformation ϕ consists in a deflection only, which means the cross-sections remain undeformed.

The plane of minimal bending rigidity, which is at the same time the plane of symmetry, determines the longitudinal section in the bar and contains the points A, M, and $\widetilde{\mathrm{M}}$. This plane will be denoted by $\prod^{L}_{(\mathrm{A},\mathrm{M},\widetilde{\mathrm{M}})}$, called *plane of deformation*, being spanned by the vectors $\widetilde{\mathbf{e}}_2$ and $\widetilde{\mathbf{e}}_3$ of the orthogonal Darboux frame $\{\widetilde{\mathrm{M}}; \widetilde{\mathbf{e}}_1, \widetilde{\mathbf{e}}_2, \widetilde{\mathbf{e}}_3\}$ associated to the middle surface $\widetilde{\mathrm{S}} = \phi$ (S) of the deformed bar, where S denotes the *middle surface* (see a mathematical definition in our book [11], p.35) of bar at the initial state. Here we have to mention that S is unique only in the case of prismatic (initially, curved or not) bars, while the round bars have before deformation an infinite number of middle surfaces consisting in all plane sections containing centroidal axis. But, during the deformation, $\widetilde{S}$ can be defined as a surface containing the deformed centroidal axis, symmetric with respect to it, of which tangent plane at each point $\widetilde{\mathrm{M}} = \phi(\mathrm{M})$ is spanned by the pair $\{ \widetilde{\mathbf{e}}_1, \widetilde{\mathbf{e}}_2 \}$ of orthogonal vectors, while $\widetilde{\mathbf{e}}_3$ is a unit normal vector to $\widetilde{S}$, tangent to the trajectory $\widetilde{\Gamma}$ of the point $\widetilde{\mathrm{M}} = \phi(\mathrm{M})$ during the deflection, that is to the *orbit* of M by the deformation ϕ. This curve is not contained by $\widetilde{S}$. But, the pair $\{ \widetilde{\Gamma}_1, \widetilde{\Gamma}_2 \}$ of curvature lines of $\widetilde{S}$ at each point $\widetilde{\mathrm{M}} = \phi(\mathrm{M})$ are assumed to be the plane curves: intersection of $\widetilde{S}$ with cross-section and the deformed centroidal axis, respectively. These curves are tangent to $\{ \widetilde{\mathbf{e}}_1, \widetilde{\mathbf{e}}_2 \}$, correspondently.

Similarly, $\prod^{T}_{\widetilde{\mathrm{M}}}$ designates the plane of cross-section of the bar through the point $\widetilde{\mathrm{M}}$; it is spanned by the vectors $\widetilde{\mathbf{e}}_1$ and $\widetilde{\mathbf{e}}_3$ of the previous frame and is tangent to the orbit $\widetilde{\Gamma}$.

The cross-sections at different points are assumed to remain undeformed during the deflection.

Let θ be the measure (in radians) of the angle $\angle(\widetilde{\mathbf{e}}_2, \mathbf{e}_2)$ at the deformed position O of the free endpoint B and ψ the similar angle at current position $\widetilde{\mathrm{M}}$.

Taking $s\,(= \lambda(\mathrm{O}\widetilde{\mathrm{M}}))$ as a parameter relative to its interval of variation, [0, L], the following initial conditions must be satisfied:

$$\psi(0) = \theta,\ \psi'(0) = 0; \quad \psi(L) = 0,\ \psi'(L) = 0 \tag{8}$$

which by a technical point of view are requested.

To render the problem amenable to a numerical treatment we achieve a “discretization” of the bar by means of a sequence of cross-section planes $\{ \prod^{T}_{\widetilde{\mathrm{M}}_i}, (i \in \overline{0,n}) \}$ trough the points $\widetilde{\mathrm{M}}_i, \left(i \in \overline{0,n}\right)$, such that $\widetilde{\mathrm{M}}_0 \equiv \mathrm{O}$ and $\widetilde{\mathrm{M}}_n \equiv \mathrm{A}$. We observe the considered above points correspond to the values

$$s_i = i \cdot \frac{L}{n}, \; (n \in N^*), \tag{9}$$

which define a division of the interval [0, L]; the norm of division is equal to $\nu = L/n$.

The exact equations of the different positions in the Euclidean affine space $\mathbf{E}^3 \equiv$(OXYZ) occupied by the deformed centroidal axis during the deflection cannot be generally known.

So, the deformed state of the bar caused by the action of bending load P can be described geometrically with the help of the system of values $\{\psi_i, \psi'_i\}_n$ at a sufficient number of points of centroidal axis, which also permit to compute the axis curvatures $(1/R)_i$ at each point $\widetilde{\mathrm{M}}_i, \left(i \in \overline{0, n}\right)$. The problem is to express them with the help of some known elements.
This is the reason of our following theorem:

Theorem 1. *The system of values* $\{\psi_i, \psi'_i\}_n$ *defining the "discrete distribution" of the deformation state of a vertical bar with a free end, discretized by a sequence of cross-section planes* $\{\prod^T_{\widetilde{\mathrm{M}}_i}, (i \in \overline{0, n})\}$, *can be estimated by recurrence formulas function only of the following positive numbers* $\theta, \mathbf{k}_0^2, \nu$, *that represent the measure of deflection angle of the bar free endpoint, the constant of physical properties, and the norm of division of the discretization, respectively. In a similar manner can be expressed the curvatures corresponding to all the points of deformed centroidal axis with respect to that discretization.*

Note. Here we have to mention that θ is a constant for a given load **P** (the charge intensity and its direction are known), in other words it depends of this last one.

Proof. Putting $\psi(s_i) = \psi_i$, ($i \in \overline{0, n}$), and using the well known approximation formulas

$$\psi'_j = \frac{\psi_{j+1} - \psi_{j-1}}{2\nu}, \quad \psi''_j = \frac{\psi_{j+1} - 2\psi_j + \psi_{j-1}}{\nu^2} \tag{10}$$

for $j \in \overline{1, n-1}$, and

$$\psi'_0 = \frac{\psi_1 - \psi_0}{\nu}, \quad \psi'_n = \frac{\psi_n - \psi_{n-1}}{\nu}, \tag{10'}$$

the initial conditions (8) lead also to the approximate values of ψ at the neighboring points of the endpoints:

$$\psi_1 = \theta, \quad \psi_{n-1} = 0, \tag{11}$$

which implies that the first and the last parts of the discretization have a similar behavior as that of the endpoints of the bar.

Also one may consider the following usual approximations:

$$\cos\,\psi_i = 1 - \frac{1}{2}\psi_i{}^2,\ \sin\psi_i = \psi_i - \frac{1}{6}\psi_i{}^3,\ \left(i \in \overline{1,n}\right). \tag{12}$$

Case II. The action of $\mathbf{P}$ is oblique.

Taking in view the first two initial conditions (8), corresponding to the point $\mathrm{O} = \phi(\mathbf{B})$, we can determine the constant C for the equation (7), as

$$C = -\mathbf{k}_0^2(\cos\theta + \frac{\delta}{c}\sin\theta),$$

such that this one becomes

$$\left(\frac{d\psi}{ds}\right)^2 = 2\mathbf{k}_0^2[\cos\psi - \cos\theta + \frac{\delta}{c}(\sin\psi - \sin\theta)]. \tag{13}$$

From here we can express the values of the derivatives along to the deformed bar with respect to the values of deflection angles at corresponding points. If we have in attention the points of chosen discretization, these values will be

$$\psi_i{}'(s) = -\sqrt{2}\,\mathbf{k}_0[\cos\psi_i - \cos\theta + \frac{\delta}{c}(\sin\psi_i - \sin\theta)]^{1/2} \tag{14}$$

at all points $\widetilde{\mathrm{M}}_{\mathrm{i}}$, $\left(i \in \overline{0,n}\right)$, where ψ_i have to be computed for this case.

On the other hand, we also can compute the coordinates (X, Y) of an arbitrary point $\widetilde{\mathrm{M}}$ of the centroidal axis of deformed bar by evaluating the curvilinear integrals

$$X = \int_0^s \cos\psi\, ds, \quad Y = \int_0^s \sin\psi\, ds,$$

if these integrals are transformed into some simple integrals with the help of arc element expressed from (13) as

$$ds = -\frac{\sqrt{2}}{2}\mathbf{k}_0^{-1}[\cos\psi - \cos\theta + \frac{\delta}{c}(\sin\psi - \sin\theta)]^{-1/2}d\psi.$$

But here we have to mention that such kind of integrals are not simple to be computed because of the complicated expression of ds. So, it remains to compute only the coordinates (X_i, Y_i) of the points $\widetilde{\mathrm{M}}_{\mathrm{i}}$, $\left(i \in \overline{0,n}\right)$, of chosen discretization by using the values ψ_i given bellow and the values (9) of s_{i}.

The initial conditions (8) are verified (for $s = 0$, and $s = L$) if and only if the following condition holds

$$tg\frac{\theta}{2} = \frac{\delta}{c}. \tag{15}$$

Moreover we also recall the validity of (11) for this case too, such that the remaining values of $\psi_k, k \in \overline{2, n-2},$ can be computed by means of some recurrence formulae as follows. First we need to express the value ψ_2 by integrating $\psi_2'(s)$ with de help of (12). So, we obtain

$$\psi_2 = \theta[1+\mathbf{k}_0^2\nu^2(\frac{\theta^2}{6}-1)]+\mathbf{k}_0^2\nu^2 tg\frac{\theta}{2}(1-\frac{\theta^2}{2}), \tag{16}$$

where $\nu = L/n$ is the norm of division. The value ψ_2 can be consider as an element of reference in order to compute the intermediate values of the function ψ at other points of the division of the real interval [0, L]. The first part of its expression,

$$A = \theta[1+\mathbf{k}_0^2\nu^2(\frac{\theta^2}{6}-1)],$$

represents the value of ψ_2 when the action of charge **P** is vertical.

In a similar way we also obtain the other values of the deflection angle at the points of discretization as

$$\psi_k = \psi_{k-2}[\frac{1}{6}\mathbf{k}_0^2\nu^2(\psi_{k-1}^2-\theta^2)+\frac{1}{\theta}A+1] - \psi_{k-2} + \psi_2 - A + \frac{1}{2}\mathbf{k}_0^2\nu^2\, tg\frac{\theta}{2}\cdot(\theta^2-\psi_{k-1}^2), \tag{17}$$

where the first part

$$B_k = \psi_{k-2}[\frac{1}{6}\mathbf{k}_0^2\nu^2(\psi_{k-1}^2-\theta^2)+\frac{1}{\theta}A+1] - \psi_{k-2}$$

represents the values of ψ at the nodes of division for $k \in \overline{2, n-2},$ as well when the action of charge **P** is vertical.

These end the proof of the first part of theorem.

Besides, the values (14) may be used to compute the curvatures of the deformed centroidal axis at the points $\widetilde{\mathrm{M}}_{\mathrm{j}}$, $\left(j \in \overline{1, n-1}\right)$:

$$(\frac{1}{R})_j = \frac{d\psi_j}{ds}, \ (j \in \overline{1, n-1}) \ . \tag{18}$$

The curvatures corresponding to endpoints of the bar are obtained using (8) and (18) as

$$(\frac{1}{R})_0 = \psi'_0 = \psi'(0) = 0, \quad (\frac{1}{R})_n = \psi'_n = \psi'(L) = 0. \tag{19}$$

But the curvatures of centroidal axis $\widetilde{\Gamma}_2$ at the points $\widetilde{\mathrm{M}}_{\mathrm{j}}, \left(j \in \overline{1, n-1}\right)$, also depend on the principal curvatures of the middle surface $\widetilde{S}$ of a curved prismatic bar. $\widetilde{S}$ is assumed undeformed, excepting a simple bending along $\widetilde{\Gamma}_2$ such that the first curvature line Γ_1 of $\widetilde{S}$ remains undeformed during bar deflection. More precisely, these values depend on the principal curvature $\widetilde{k}_2$ according to the fact that $\widetilde{\Gamma}_2$ is also a curvature line of $\widetilde{S}$. So, we may use the formula (see Boja, Ivan, Brailoiu, [1987])

$$\widetilde{k}_2 = \frac{1}{\sqrt{\widetilde{g}_2}} \cdot (\frac{1}{R})_2, \quad (\widetilde{g}_2 = \langle \widetilde{\mathbf{e}}_2 \, ; \, \widetilde{\mathbf{e}}_2 \rangle) \tag{20}$$

in order to compute the curvatures $(\widetilde{k}_2)_{\mathrm{i}}$ of $\widetilde{S}$ at all points $\widetilde{\mathrm{M}}_{\mathrm{i}}, \left(i \in \overline{0, n}\right)$.
This ends the proof. #

3. Approximate and exact formulas of the bar bending energy

In two papers published before (see Boja, Ivan, Brailoiu, [1993] and Brailoiu, Boja, [1993]) the following results were established:

Theorem 2. *The strain energy of deformation (under a pure bending moment M) of a slender bar with a fixed endpoint, loaded with a charge P and acting oblique at the free endpoint, is given by the exact formula*

$$U_M = P \cdot \int_0^L [\cos\psi - \cos\theta + tg\frac{\theta}{2}(\sin\psi - \sin\theta)]\cos\psi \cdot ds, \tag{21}$$

where L denotes the length of the bar, ψ is defined by the centroidal axis equation (1), and $\theta = \psi(0)$ by an initial condition is given. ##

Starting with a discretization of the bar with the help of the family of planes $\{ \prod_{\widetilde{\mathrm{M}}_{\mathrm{i}}}^{T}, (i \in \overline{0, n}) \}$ orthogonal to the tangent straight lines to the centroidal axis at points that correspond to the sequence of arc values (9) one obtain the following formulae for the energy of deformation

$$\Delta U = P\frac{L}{n} \cdot [\sum_{i=1}^{n} (\cos\, \psi_i - \cos\, \theta) \cos\, \psi_i + tg\frac{\theta}{2} \sum_{i=1}^{n} \, (\sin \psi_i - \sin \theta) \cos\, \psi_i], \tag{22}$$

where also can be used the approximations (12).

This corresponds to the case I, when $\mathbf{P}$ acts vertically. So, only its measure P is involved.

The values (21) and (22) can be evaluated when in (12) all the values $\psi(s_i) = \psi_i$, ($i \in \overline{1, n}$), are known. Thus, if we make use of the constant $\mathbf{k}_0^{\,2}$ introduced by (6), one can obtains by recurrence the asked before values. We observe the values (11) do not depend on $\mathbf{k}_0$ because of the assumption that at the neighbor points to endpoints of the centroidal axis
the approximation can be considered alike that at the bar extremities.

The vertical displacement of the free end

$$\lambda = \frac{1}{2}\int_0^L tg^2\psi \cdot cos\ \psi\, ds \tag{23}$$

can be used to compute the work of the charge P, $\Delta T = P \cdot \lambda$. Thus, for a discretization as that considered before, we have

$$\Delta T = \frac{P \cdot L}{2n} \cdot \sum_{i=1}^{n} tg^2\psi_i \cdot cos\,\psi_i\ . \tag{24}$$

We remark the expression (24) can not be used to determine critical value of P.

4. Numerical example

Consider a bar with the physical and geometrical characteristics given below.

To solve the problem exposed in the Sections 2 and 3 we used a FreeFem++ soft.

```
//*** Physical and geometrical constants of the material : Iron ***
real L=1.60;                    //m; length of the bar
real E=210*1e+009;             //  N*m^(-2)
real a=2*1e-002;               //  m ; length of an edge of the rectangular cross section
real b=5*1e-002;               //  m ; length of another edge of the rectangular cross section
real vol=a*b*L;                //  volume of the bar
real ro=7874;                  //  kg/m^3
real mass =vol*ro;             //  mass of the bar
int n=24;                      //  nr of the nodal points
real niu=L/n;                  // norm of the division
real P=6000;                   // N; load of the bar
real I=masa*(a^2+b^2)/12;  // inertial moment
real k0=(P/(E*I))^(1/2);
real theta0=1;                 // radians
//************* Finite elements' space ***************
mesh Th=square(5,2);
fespace Vh(Th,P2);
real[int] psi(n);
real[int] xx(n), yy(n);
psi[0]=theta0;
psi[1]=theta0;
psi[2]=theta0*(1+(k0^2)*(niu^2)*((theta0^2)/6-1));
psi[n-1]=0;
int i;
for (i=3;i<n-1;i++)
{psi[i]=psi[i-1] * (1+(1/6)*(k0^2)*(niu^2)*((psi[i-1])^2-theta0^2)+(1/theta0)*psi[2]) - psi[i-2];};
```

```
for (i=0;i<n;i++)
{xx[i]=i*niu;  yy[i]=psi[i];  cout << i << " " << yy[i] << "\n";};
plot([xx,yy], cmm =" grafic: theta= "+theta0+ " lungimea barei = "+L, wait=1, nbiso=20, fill=1, value=1,
ps="lucrare2.eps");
```

4.1. Table of the values so obtained in the case of the bar loaded oblique:

Node „i”	s_i	ψ_i	Node „i”	s_i	ψ_i
0	0	1.3	12	0.8	1.06881
1	0.0666667	1.3	13	0.866667	1.02853
2	0.133333	1.29637	14	0.933333	0.985484
3	0.2	1.28913	15	1	0.93982
4	0.266667	1.2783	16	1.06667	0.891679
5	0.333333	1.26391	17	1.13333	0.841214
6	0.4	1.246	18	1.2	0.788588
7	0.466667	1.22465	19	1.26667	0.733968
8	0.533333	1.1999	20	1.33333	0.677528
9	0.6	1.17185	21	1.4	0.619449
10	0.666667	1.14059	22	1.46667	0.559916
11	0.733333	1.1062	23	1.53333	0

References

[1]. H. H. Denman, , R Schmidt. An approximate method of analysis of large deflections; *J. of Appl. Math and Phys.*, vol.21, **3**, 412 - 421, Basel , 1970.

[2]. R.,Schmidt, D. A. Dadeppo. Large deflections of heavy cantilever beams and columns; *Quart. Appl. Math*., 28, **3**, 441- 444, 1970.

[3]. M. C. Dokmeci . A general theory of elastic beams; *Int. J. Solids Struct*., **8**, 1205 - 1222, 1972.

[4]. A. Pisanty. Geometrically nonlinear behavior of initially curved plane slender bars and frames; *Ing.-Archiv*., **46**, 233 - 244, 1977.

[5]. N. Boja., M. Ivan., G. Brailoiu. Sur une expression geometrique de l'etat deformé pour les barres a la section courbe; *Proceed. of the 2nd Symp. Math.and its Appl*., Timisoara, 282 - 286, 1987.

[6]. N. Boja., G. Brailoiu, R. Ene. Some approximations in the bar deflection analysis,I.; *Bull.for Appl.and Comp.Mathematics*, BAM- CXIII, #2382, Budapest, 001 - 008, 2008.

[7]. I. Bors, P. Alexa. Elastica of a beam taking into account the axial and shear effects; *Stability of Steel Structures*, Akad. Kiadó, Budapest, vol.**2**, 1291 - 1296, 1990.

[8]. N. Boja., M. Ivan., G. Brailoiu. Sur l'énergie de déformation des barres a section courbe discretisées; *Proceed. of the 5nd Symp. Math.and its Appl*., Timisoara, 32 - 39, 1993.

[9]. L. Trabucho, I. M. Viaño. *Mathematical Modelling of Rods*, Handbook of Numerical Analysis; Elsevier

Sci. Publ. B.V., North-Holland, 1993.

[10]. G. Brailoiu, N. Boja. Sur l'énergie de déformation des barres discretisées sous l'action des forces passant par un point fix*; Bul. St. Teh., Univ.Tech.Timisoara, Ser. Mat.-Fiz.*, Tom 38(52), 45-50, (1993).

[11]. N. Boja., G. Brailoiu. *Geometric Approach to the Continuum Deformation Analysis*; Monograph. Book. Series , MB-**3** / PAMM, Techn. Univ. Budapest, 1997.

[12]. M. Epstein, M. Elzanowski. *Material Inhomogeneities and their Evolution - A Geometric Approach*; Springer- Verlag, Berlin – Heidelberg - New York, 2007.

[13]. D. Lanc, G. Turkalj, G. Vizentin . An algorithm for numerical creep buckling analysis of beam-type structures*; Bull. For Appl. and Comp. Math.*, vol. BAM – CX , #2302, Beograd-Budapest, 55 - 60, 2007.

NICOLAE BOJA
"Politehnica" University Timisoara
Department of Mathematics
Victoriei Sq.2
ROMANIA
E-mail: nicolae.boja@yahoo.com

GEORGETA BRAILOIU
"Politehnica" University Timisoara
Department of Mathematics
Victoriei Sq.2
ROMANIA
E-mail: -

REMUS ENE
"Politehnica" University Timisoara
Department of Mathematics
Victoriei Sq.2
ROMANIA
E-mail: eneremus@gmail.com