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From Bernstein Polynomials to Lagrange Interpolation

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Abstract

For a given continuous function $f(x)$ on $[0,1]$ we construct sequence of algebraic polynomials based on Bernstein approximation. We prove that the limit of this sequence is the Lagrange interpolation polynomial of degree n. Application to the representation of polynomial curves will be given.

1 Introduction

For any continuous function $f \in C[0,1]$ the classical $n-$th degree Bernstein polynomial at the point $x \in [0,1]$ is given by

$$B_n(f;x) = \sum_{k=0}^{n} f(\frac{k}{n}) \cdot p_{n,k}(x), \tag{1.1}$$

where the Bernstein basis polynomials $p_{n,k}$ are defined as

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}. \tag{1.2}$$

Setting

$$\bar{F} := \begin{pmatrix} f(0) \\ f(\frac{1}{n}) \\ \vdots \\ f(1) \end{pmatrix} = (f(0), f(\frac{1}{n}), \ldots, f(1))^t, \tag{1.3}$$

and

$$\bar{b}_n(x) = (p_{n,0}(x), p_{n,1}(x), \ldots, p_{n,n}(x)) \tag{1.4}$$

we may rewrite (1.1) in matrix representation by

$$B_n(f,x) = \bar{b}_n(x) * \bar{F}, \tag{1.5}$$

where in the remainder of the paper $*$ always denotes matrix multiplication. The Lagrange interpolation polynomial (Lagrange interpolant) of degree n is given by

$$L_n(f,x) = \sum_{k=0}^{n} f(\frac{k}{n}) \cdot l_{n,k}(x), \tag{1.6}$$

where the Lagrange basis polynomials $l_{n,k}$ are defined as

$$l_{n,k}(x_i) = \delta_{i,k} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}, \; 0 \leq i, k \leq n. \tag{1.7}$$

Setting

$$\bar{l}_n(x) = (l_{n,0}(x), l_{n,1}(x), \dots, l_{n,n}(x)) \tag{1.8}$$

we may rewrite (1.6) by

$$L_n(f,x) = \bar{l}_n(x) * \bar{F}. \tag{1.9}$$

The essential role in our further considerations plays the following $(n+1)\times(n+1)$ matrix T, whose elements $[t_{k,j}^{(n)}]$, $k,j = 0,\dots,n$ are the values of Bernstein basis polynomials $\{p_{n,i}(x)\}_{i=0}^{n}$ at the knots $0, \frac{1}{n}, \frac{2}{n}, \dots, 1$:

$$T := \begin{pmatrix} p_{n,0}(0) & p_{n,1}(0) & p_{n,2}(0) & \dots & p_{n,n}(0) \\ p_{n,0}(\frac{1}{n}) & p_{n,1}(\frac{1}{n}) & p_{n,2}(\frac{1}{n}) & \dots & p_{n,n}(\frac{1}{n}) \\ \dots & \dots & \dots & \dots & \dots \\ p_{n,0}(1) & p_{n,1}(1) & p_{n,2}(1) & \dots & p_{n,n}(1) \end{pmatrix} \tag{1.10}$$

This matrix was studied and applied in many different areas of analysis, numerical methods, computer aided geometric design etc. For example in [1] a new fast method was introduced, to approximate the value of a definite integral of $f \in C[0,1]$. This method gives considerable better results for a broad class of sufficiently differentiable functions, if compared with other known quadrature rules like Simpson rule, composite trapezoid rule etc. The author has not observed that the limiting operator defined in [1] by $J_n(f)$ is actually the Lagrange interpolant of degree n, defined in (1.6). The algorithm developed in [1] is the main motivation for us to write this paper. Other application of the matrix T is given in [4] to construct recursive subdivision algorithm for polynomial curves-one of the basic tools in computer aided geometric design (CAGD). In [4] it was shown, that the control polygons produced by recursive subdivision always converge to their original curve. We may continue with many other applications of the matrix T, but these two examples are enough to show its significance and application to solve different problems. In our paper we consider only the values of a given function f at equidistant knots $\frac{k}{n}$, $k = 0,1,\dots,n$. This may be extended to arbitrary set of knots $0 \le t_0 < t_1 < \dots < t_n \le 1$ as it was studied in [4]. Our main result states the following

Theorem 1 *If f is any bounded function, defined on the interval $[0,1]$, then for all $n = 1,2,\dots$ and all $x \in [0,1]$ we have*

$$\bar{l}_n(x) * \bar{F} = \bar{b}_n(x) * T^{-1} * \bar{F}, \tag{1.11}$$

where T^{-1} is the inverse matrix of T.

In Section 2 we establish some properties of T, we construct a sequence of algebraic polynomials $\{\tilde{B}_m\}_{m=1}^{\infty}$ of degree n and show that its limit is the Lagrange interpolant $L_n(f)$. The proof of Theorem 1 is based on this observation. In Section 3 we show some applications and corollaries of our main result.

2 Proof of Theorem 1

We establish some properties of the matrix T.

Lemma 1 *The matrix T is nonsingular.*

Proof: If we suppose the contrary, then the columns are linear dependent, i.e. there are constants $\lambda_0, \lambda_1, \dots, \lambda_n$, at least one of which different from 0, such that

$$h_n(x) := \lambda_0 \cdot p_{n,0}(x) + \lambda_1 \cdot p_{n,1}(x) + \dots + \lambda_n \cdot p_{n,n}(x) = 0$$

holds for $x = 0, \frac{1}{n}, \frac{2}{n}, \dots, 1$. Hence $h_n(x) \equiv 0$ for all $x \in [0,1]$. It follows that polynomials $\{p_{n,i}(x)\}_{i=0}^{n}$ should be linear dependent, but this is not possible, because they build basis in the space of all algebraic polynomials of degree $\le n$. Therefore P is a nonsingular matrix. □

The matrix T was used also in [2] to establish the eigenstructure of the classical Bernstein operator. An useful property of T is that T is a positive definite -see Proposition 5.1 in [5]. Using this fact the following was proved in Lemma 4.3 in [1], which we formulate here as

Lemma 2 *For $i = 0,1,\dots,n$ we have $0 < \lambda_i(T) \le 1$, where $\lambda_i(T)$ are the eigenvalues of T.*

Lemma 2 implies that $\rho(I-T)<1$ where $\rho(I-T)$ is the spectral radius of the matrix $I-T$, with I-the identity matrix. In [2] it was established that the eigenvalues of the operator B_n are given by

$$\lambda_k^{(n)} := \frac{n!}{(n-k)!n^k},\ k=0,1,\ldots,n.$$

Therefore

$$1=\lambda_0^{(n)}=\lambda_1^{(n)}>\lambda_2^{(n)}>\ldots>\lambda_n^{(n)}>0.$$

If $f_k(x)$ is the eigenfunction of B_n corresponding to $\lambda_k^{(n)}$ then

$$B_n(f_k,x)=\lambda_k^{(n)}\cdot f_k(x). \tag{2.1}$$

Consequently we arrive at

Lemma 3 *The eigenvalues of T choincides with the eigenvalues of B_n.*

Proof: In the representation (2.1) we set $x=0,\frac{1}{n},\frac{2}{n},\ldots,1$ and obtain

$$T*\bar{f}_k=\lambda_k^{(n)}\cdot I*\bar{f}_k,$$

where $\bar{f}_k=[f_k(0),f_k(\frac{1}{n}),\ldots,f_k(1)]^t$. The last equation implies that

$$\det\left(T-\lambda_k^{(n)}\cdot I\right)=0,$$

i.e. $\lambda_k^{(n)}$ is an eigenvalue of T. □

The next statement is well known from the theory of linear algebra, namely

Lemma 4 *If A is a square matrix with $\rho(A)<1$ then the matrix $I-A$ is invertible and we have*

$$(I-A)^{-1}=I+A+A^2+\ldots.$$

If we set $A:=I-T$ in the last formula we obtain

Lemma 5 *The matrix T is invertible and we have*

$$T^{-1}=I+(I-T)+(I-T)^2+\ldots=\prod_{m=0}^{\infty}(I-T)^m. \tag{2.2}$$

Lemma 5 gives another proof of the property of T formulated in Lemma 1 and in addition the useful representation of the inverse matrix T^{-1}. We end our study of T with the following observation (see Lemma 2.2 in [4]):

Lemma 6 *All rows of T and T^{-1} sum to 1.*

Let us calculate T explicitly for $n=1,2,3$

$$n=1\ T=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ T^{-1}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$n=2\qquad T=\begin{pmatrix}1&0&0\\\frac14&\frac12&\frac14\\0&0&1\end{pmatrix},\ T^{-1}=\begin{pmatrix}1&0&0\\-\frac12&2&-\frac12\\0&0&1\end{pmatrix}$$

$$n=3\qquad T=\begin{pmatrix}1&0&0&0\\\frac{8}{27}&\frac49&\frac29&\frac{1}{27}\\\frac{1}{27}&\frac29&\frac49&\frac{8}{27}\\0&0&0&1\end{pmatrix},\ T^{-1}=\begin{pmatrix}1&0&0&0\\-\frac56&3&-\frac32&\frac13\\\frac13&-\frac32&3&-\frac56\\0&0&0&1\end{pmatrix}.$$

Let f be arbitrary bounded function defined on $[0,1]$ and let $x \in [0,1]$-be fixed. We may write recursively

$$\begin{aligned}
&f(x) = B_n(f,x) + r_1(x)\\
&r_1(x) = B_n(r_1,x) + r_2(x)\\
&r_2(x) = B_n(r_2,x) + r_3(x)\\
&\ldots\\
&r_m(x) = B_n(r_m,x) + r_{m+1}(x).
\end{aligned} \tag{2.3}$$

It is easy to verify, that by the method introduced in (2.3) we construct a sequence of n-th degree algebraic polynomials $\{\tilde{B}_m(f,x)\}_{m=0}^{\infty}$ defined by

$$\tilde{B}_m(f,x) = \bar{b}_n(x) * \left[\sum_{i=0}^{m}(I-T)^i\right] * \bar{F}, \tag{2.4}$$

which satisfy

$$f(x) = \tilde{B}_m(f,x) + r_{m+1}(x),\ m = 0,1,2,\ldots \tag{2.5}$$

It is easy to observe that

$$\begin{aligned}
&\tilde{B}_0(f,x) = B_n(f,x)\\
&\tilde{B}_1(f,x) = B_n(f,x) + \bar{b}_n(x) * (I-T) * \bar{F} =\\
&= B_n(f,x) + B_n(f,x) - B_n \circ B_n(f,x) =\\
&= B_n(f,x) + B_n(f,x) - B_n^2(f,x) =\\
&= B_n(f,x) - B_n(f - B_n f, x),
\end{aligned} \tag{2.6}$$

where $B_n^k := B_n \circ B_n \circ \ldots \circ B_n\ -k$ times is the k-th iterates of the Bernstein operator. By the definition of the polynomial sequence $\{\tilde{B}_m\}_{m=0}^{\infty}$ in (2.4) we may consider $\tilde{B}_m$ as linear operator $\tilde{B}_m : C[0,1] \to C[0,1]$. It is clear from (2.6) that $\tilde{B}_m$ are not positive operators, like the Bernstein operator B_n. On the other hand they interpolate the function f at the ends and it is natural to expect that $\tilde{B}_m f$ approximates f at the point x better than $B_n f$. For example if $f = e_2 : x \to x^2$, $x \in [0,1]$ it is known that (see Ch. 10 in [3])

$$\begin{aligned}
&\tilde{B}_0(e_2,x) = B_n(e_2,x) = e_2(x) + \tfrac{x(1-x)}{n},\\
&\tilde{B}_1(e_2,x) = e_2(x) + \tfrac{x(1-x)}{n^2}, \ldots\\
&\tilde{B}_m(e_2,x) = e_2(x) + \tfrac{x(1-x)}{n^{m+1}}.
\end{aligned} \tag{2.7}$$

It is clear from (2.7) that the error of approximation of $e_2(x)$ by $\tilde{B}_m(e_2,x)$ is essentially smaller than by $B_n(e_2,x)$. It is known that the optimal rate of approximation for B_n is $O(\frac{1}{n})$, $n \to \infty$ and if $f(x) - B_n(f,x) = o_x(\frac{1}{n})$, $n \to \infty$, then f is linear function -see Theorem 5.1 in Ch. 10 in [3]. It is easy to observe that for all $m \geq 0$, $\tilde{B}_m$ preserves linear functions. So we may conclude that the loss of positivity is compensate for by better degree of approximation. Consequently from (2.1) and (2.4) we get

$$\tilde{B}_\infty(f,x) = \bar{b}_n(x) * T^{-1} * \bar{F}. \tag{2.8}$$

Lemma 7 *The sequence of polynomials $\{\tilde{B}_m f\}_{m=0}^{\infty}$ uniformly tends to its limiting operator $\tilde{B}_\infty f$ over the interval $[0,1]$ when $m \to \infty$.*

Proof: The representations (2.4) and (2.8) imply

$$\tilde{B}_\infty(f,x) - \tilde{B}_m(f,x) = \bar{b}_n(x) * \left[\sum_{i=m+1}^{\infty}(I-T)^i\right] * \bar{F}. \tag{2.9}$$

By Lemma 5 we know that the power series (2.2) is convergent and this implies that the matrix $[\sum\limits_{i=m+1}^{\infty}(I-T)^i] \to 0$ when $m \to \infty$. This completes the proof. $\square$

By elementary calculations we obtain that if $n = 2$ the limiting operator $\tilde{B}_\infty f$ preserves the monomial functions e_i, $i = 0, 1, 2$ and if $n = 3$ then $\tilde{B}_\infty f$ preserves all e_i, $i = 0, 1, 2, 3$. This could be generalized for arbitrary natural number n. Theorem 1 implies that the operator $\tilde{B}_\infty f$ reproduces all algebraic polynomials of degree not greater than n.

Proof of Theorem 1

For a given function f let us denote by $\bar{B}_n f$ the vector

$$\bar{B}_n f := [B_n(f, 0), B_n(f, \frac{1}{n}), \dots, B_n(f, 1)]^t. \tag{2.10}$$

It is clear that from (1.3) and the definition of T we get

$$\bar{B}_n f = T * \bar{F}. \tag{2.11}$$

In a similar way if we denote by

$$\bar{B}_\infty := [\tilde{B}_\infty(f, 0), \tilde{B}_\infty(f, \frac{1}{n}), \dots, \tilde{B}_\infty(f, 1)]^t$$

then (2.8) and (2.11) imply

$$\bar{B}_\infty = T * T^{-1} * \bar{F} = I * \bar{F} = \bar{F}, \tag{2.12}$$

that is the n-th degree algebraic polynomial $\tilde{B}_\infty(f, x)$ interpolates the function f at the knots $0, \frac{1}{n}, \frac{2}{n}, \dots, 1$. On the other hand such an algebraic polynomial is uniquely defined and this is n-th degree Lagrange interpolant $L_n(f, x)$ for a given function f. Therefore we arrive at

$$\tilde{B}_\infty(f, x) \equiv L_n(f, x).$$

The proof of Theorem 1 is completed. $\square$

3 Applications

A. Approximation properties

If $f \in C^{n+1}[0, 1]$ then the remainder for Lagrange interpolant can be represented as

$$f(x) = L_n(f, x) + \frac{(x-0)(x-\frac{1}{n})\cdots(x-1)}{(n+1)!} \cdot f^{(n+1)}(), \tag{3.1}$$

for some $\in (0, 1)$. This implies

Theorem 2 *If $f \in C^{n+1}$ and $\|f^{(n+1)}\| \le M_{n+1}$, where $\|\cdot\|$ denotes sup norm, then*

$$\|f - L_n f\| \le \frac{M_{n+1}}{(n+1)!} \to 0,\ n \to \infty. \tag{3.2}$$

Also using (2.9) and Lemma 3 we can easily prove

Theorem 3 *For $f \in C[0, 1]$ the following holds true*

$$\|\tilde{B}_\infty f - \tilde{B}_m f\| \le \|f\| \cdot (1 - \lambda_n^{(n)})^{m+1}. \tag{3.3}$$

Therefore

$$\lim_{m\to\infty} \|\tilde{B}_\infty f - \tilde{B}_m f\| = 0.$$

B. Applications in CAGD

Let us consider the vector-valued parametric-defined function $f : [0,1] \to R^d$, $d \geq 2$. If f is $n-$th degree algebraic polynomial, then as usual we associate with $f(x)$ the d-dimensional polynomial curve $\vec{C}(x)$, namely

$$\vec{C}(x) = f(x),\, x \in [0,1].$$

It is known from CAGD that any polynomial curve $\vec{C}(x)$ may be represented in its Bernstein-Bézier form as

$$\vec{C}(x) = \bar{b}_n(x) * \bar{P}, \tag{3.4}$$

where

$$\bar{P} := [P_0, P_1, \ldots, P_n]^t,\, P_i \in R^d,\, 0 \leq i \leq n,$$

denotes the control polygon of the curve and P_i are the control points of $\vec{C}(x)$. From Theorem 1 we immediately get

Corollary 1 *For any polynomial curve $\vec{C}(x)$, $x \in [0,1]$ the control points in its Bernstein-Bézier representation can be computed by*

$$\bar{P} = T^{-1} * \bar{F}, \tag{3.5}$$

where $\bar{F}$ is defined in (1.3) *and consists of points, lying on the curve.*

Our next statement is inverse to the previous one. If now the vector of control points $\bar{P}$ is given, then we can easily compute the coordinates of the points $\vec{C}(\frac{k}{n})$, $0 \leq k \leq n$. Hence

Corollary 2 *Let the curve $\vec{C}$ be defined by* (3.4). *Then the vector $\bar{F}$ can be computed by*

$$\bar{F} = T * \bar{P}. \tag{3.6}$$

In [4] a subdivision algorithm was introduced, such that the control polygons, obtained in each step, uniformly tend to the curve. Similar statement follows immediately from Theorem 1 and the sequence of polynomials $\tilde{B}_m(f,x)$ defined in (2.4)

Corollary 3 *The sequence of control polygons of the curves, associated with the polynomials $\tilde{B}_m(f,x)$ uniformly w.r.t. $x \in [0,1]$ tends to the control polygon of the curve $\vec{C}(x) = f(x)$.*

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