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On some fuzzy positive and linear operators

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Abstract

The purpose of this work is to show that fuzzy Bernstein-Stancu operators introduced in [3] satisfy an A-statistical version of fuzzy Korovkin theorem. Some properties of these operators are also proved. An example of new fuzzy positive and linear operators is presented.

1 Introduction

Sometimes, the phenomena encountered in real life do not have a precise definition for membership criteria. For example, the class of plants obviously includes flowers, trees, grass but there are elements like amoeba or bacteria which have an ambiguous status regarding their belonging to plants class.

Similarly, we are dealing with ambiguity when we want to compare the number 30 with the class of real numbers much greater than zero, which obviously do not have a precise definition. In mathematics, this kind of uncertainty can be modeled in two ways. First way is a probabilistic one and the second way refers to fuzzy logic.

For the first time modeling uncertainty through fuzzy logic was approached by L.Zadeh [13] who introduced the fuzzy sets as basis for reasoning with multiple truth values. Later on, the concept was applied in many areas of science like finance, weather prediction, hand writing analysis, electronics, biomedicine or elevators.
Considering the wide applicability of fuzzy sets naturally arises the fuzzy functions approximation problem (which can model complicate real processes) with effectively calculable functions.

In this paper, starting from classical Bernstein-Stancu operator defined in [10],

$$(B_m^{(\alpha,\beta)}f)(x)=\sum_{k=0}^{m}\binom{m}{k}x^k(1-x)^{m-k}f\left(\frac{k+\alpha}{m+\beta}\right),m\in\mathbb{N},x\in[0,1],$$

and considering its fuzzy variant defined in [3], we prove some properties concerning this variant and we also introduce a new class of fuzzy operators.

In Section 2, we collect some basic elements used throughout the paper. Further on, we prove that the fuzzy Bernstein-Stancu operators satisfy a fuzzy Korovkin - type theorem and this theorem holds for our new class of operators.

2 Preliminaries

We need the following definitions.

Definition 1 ([*12*]) *Let* $\mu:\mathbb{R}\to[0,1]$ *with the following properties:*

(i) *μ is normal, i.e,* $\exists\, x_0 \in \mathbb{R},\ \mu(x_0) = 1$;

(ii) *μ is a convex fuzzy subset, i.e,* $\forall x, y \in \mathbb{R}, \forall \lambda \in [0,1],\ \mu(\lambda x + (1-\lambda)y) \geq \min\{\mu(x), \mu(x)\}$;

(iii) *μ is upper semicontinuous on $\mathbb{R}$, i.e, $\forall x_0 \in \mathbb{R}$ and $\forall \epsilon > 0$, a neighborhood $V(x_0)$ of x_0 exists such that* $\mu(x) \leq \mu(x_0) + \epsilon,\ \forall x \in V(x_0)$;

(iv) *The set $\overline{\{x \in \mathbb{R} : \mu(x) > 0\}}$ is compact in $\mathbb{R}$, where $\bar{A}$ denotes the closure of A.*

The function μ is called fuzzy number.

The set of all μ is denoted by $\mathbb{R}_{\mathcal{F}}$.
For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define

$$[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\} \quad \text{and} \quad [\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) > 0\}}.$$

We recall now that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded real interval, see [8].

Definition 2 ([12]) *Let $u, v \in \mathbb{R}$ and let $\lambda \in \mathbb{R}$. We define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ as follows*

$$[u \oplus v]^r = [u]^r + [v]^r, \qquad [\lambda \odot u]^r = \lambda [u]^r, \quad \forall r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda[u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$.

Notice that $1 \odot u = u$ and $u \oplus v = v \oplus u,\ \lambda \odot u = u \odot \lambda$ hold.

Remark 1 *If we have $0 \leq r_1 \leq r_2 \leq 1$, then $[u]^{r_1} \subseteq [u]^{r_2}$. We denote the interval $[u]^r = [u_-^{(r)}, u_+^{(r)}]$, where $u_-^{(r)} \leq u_+^{(r)},\ \ u_-^{(r)}, u_+^{(r)} \in \mathbb{R}_+,\ \forall r \in [0,1]$.*

In the following sections we will use the fuzzy metric given in [11],

Definition 3 *Let $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+$ be defined as follows*

$$\begin{aligned} D(u,v) &:= \sup_{r \in [0,1]} \max\{|u_-^{(r)} - v_-^{(r)}|, |u_+^{(r)} - v_+^{(r)}|\} \\ &= \sup_{r \in [0,1]} \rho([u]^r, [v]^r), \end{aligned}$$

where $[v]^r = [v_-^{(r)}, v_+^{(r)}]$, $u, v \in \mathbb{R}_{\mathcal{F}}$ and ρ is the Hausdorff distance.

Following [11], $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space.
Let $f, g : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be fuzzy number valued functions. Then, according to [5], the distance between f and g is given by

$$D^*(f,g) := \sup_{x \in [a,b]} \sup_{r \in [0,1]} \max\{|f_-^{(r)}(x) - g_-^{(r)}(x)|, |f_+^{(r)}(x) - g_+^{(r)}(x)|\}.$$

Definition 4 ([2], [7]) *Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy real number valued function. We define the first fuzzy modulus of continuity of f by*

$$\omega_1^{(\mathcal{F})}(f, \delta) := \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} D(f(x), g(x)), \qquad \delta \in (0, b-a].$$

Definition 5 ([3]) *Let $f \in C([0,1], \mathbb{R}_{\mathcal{F}})$. We define the second fuzzy modulus of continuity of f by*

$$\omega_2^{(\mathcal{F})}(f,h) := \sup_{\substack{u,v \in [0,1] \\ |u-v| \leq 2h \\ h > 0}} \left\{ D\left(f(u) \oplus f(v), 2 \odot f\left(\frac{u+v}{2}\right)\right)\right\}.$$

The set of all fuzzy continuous functions on the interval $[a, b]$ is denoted by $C_{\mathcal{F}}[a, b]$.

Remark 2 ([2]) *Let $f, g \in C_{\mathcal{F}}[a, b]$. We say that f is fuzzy larger than g pointwise and we denote it by $f \succeq g$ if and only if*

$$f(x) \succeq g(x) \quad \text{iff} \quad f_-^{(r)}(x) \geq g_-^{(r)}(x) \quad \text{and} \quad f_+^{(r)}(x) \geq g_+^{(r)}(x),\ \forall x \in [a, b],\ \forall r \in [0, 1].$$

Definition 6 ([5]) *Let $L : C_{\mathcal{F}}[a, b] \to C_{\mathcal{F}}[a, b]$ be an operator. Then L is said to be fuzzy linear if, for every $\lambda_1, \lambda_2 \in \mathbb{R},\ f_1, f_2 \in C_{\mathcal{F}}[a, b]$, and $x \in [a, b]$,*

$$L(\lambda_1 \odot f_1 \oplus \lambda_2 \odot f_2; x) = \lambda_1 \odot L(f_1; x) \oplus \lambda_2 \odot L(f_2; x)$$

holds. Also L is called fuzzy positive linear operator if it is fuzzy linear and the condition $L(f; x) \preceq L(g; x)$is satisfied for any $f, g \in C_{\mathcal{F}}[a, b]$ and all $x \in [a, b]$ with $f(x) \preceq g(x)$.

We now recall the definition of fuzzy Bernstein-Stancu operators as it was given in [3].

Definition 7 *Let $f \in C([0, 1], \mathbb{R}_{\mathcal{F}}),\ m \in \mathbb{N},\ 0 \leq \alpha \leq \beta$. We define*

$$({}^{\mathcal{F}}L_m^{\alpha,\beta} f)(x) = \sum_{k=0}^{m}{}^{*} p_{m,k}(x) \odot f\left(\frac{k + \alpha}{m + \beta}\right), \quad x \in [0, 1], \tag{1}$$

where $p_{m,k}(x) = \begin{pmatrix} m \\ k \end{pmatrix} x^k (1 - x)^{m-k}$.

Here $\sum^*$ stands for fuzzy summation.

For the reader's convenience, we recall the almost convergence, the statistically convergence, the A-statistically convergence of a real sequence, and the fuzzy Korovkin theorem.

Based on the result of Lorentz [9], a bounded real sequence $(x_n)_{n\in\mathbb{N}}$ is said to be almost convergent to a real number L if and only if

$$\lim_{p->\infty} \frac{x_n + ... + x_{n+p-1}}{p} = L.$$

Definition 8 ([6]) *A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is called statistically convergent to a real number L, if for every $\varepsilon > 0$,*

$$\delta(\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}) = 0,$$

where

$$\delta(E) := \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{n} \chi_E(j)$$

represents the density of the set $E \subseteq \mathbb{N}$, and χ_E is the characteristic function associated to set E. We denote this limit by

$$st - \lim_n x_n = L.$$

Definition 9 *Let $A = (a_{j,n})_{j,n\in\mathbb{N}}$ be a non-negative regular summability method. The sequence $(x_n)_{n\in\mathbb{N}}$ is said to be A-statistically convergent to a real number L if for every $\varepsilon > 0$,*

$$\delta_A(\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}) = 0,$$

or equivalent

$$\lim_{j\to\infty} \sum_{\{n:|x_n - L| \geq \varepsilon\}} a_{j,n} = 0.$$

We denote this limit by

$$st_A - \lim_n x_n = L.$$

We now give the fuzzy Korovkin theorem.

Theorem 1 ([2]) *Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of fuzzy positive linear operators from $C_{\mathcal{F}}[a,b]$ into itself. Assume that there exists a corresponding sequence $(\widetilde{L_n})_{n\in\mathbb{N}}$ of positive linear operators from $C[a,b]$ into itself with the property*

$$\{L_n(f;x)\}_{\pm}^{(r)} = \widetilde{L_n}(f_{\pm}^{(r)};x) \tag{2}$$

for all $x\in[a,b]$, $r\in[0,1]$, $n\in\mathbb{N}$ and $f\in C_{\mathcal{F}}[a,b]$. Assume further that

$$\lim_n \|\widetilde{L_n}(e_i)-e_i\| = 0 \quad \text{for each} \quad i=0,1,2.$$

Then, for all $f\in C_{\mathcal{F}}[a,b]$, we have

$$\lim_n D^*(L_n(f),f)=0.$$

3 Main results

In this section we prove that fuzzy Bernstein-Stancu operators satisfy the A-statistical version of fuzzy Korovkin theorem which was first obtained in [2].

First of all, we give the following lemma:

Lemma 1 *The fuzzy Bernstein-Stancu operators defined by* (1) *are positive and linear operators.*

Proof. The linearity:

$$\begin{aligned}
{}^{\mathcal{F}}L_m^{\alpha,\beta}(\lambda_1\odot f_1\oplus\lambda_2\odot f_2;x) &= \sum_{k=0}^{m}{}^{*} p_{m,k}(x)\odot\left[\lambda_1\odot f_1\left(\frac{k+\alpha}{m+\beta}\right)\oplus\lambda_2\odot f_2\left(\frac{k+\alpha}{m+\beta}\right)\right]\\
&= \lambda_1\underbrace{\sum_{k=0}^{m}{}^{*} p_{m,k}(x)\odot f_1\left(\frac{k+\alpha}{m+\beta}\right)}_{{}^{\mathcal{F}}L_m^{(\alpha,\beta)}(f_1;x)}\oplus\lambda_2\underbrace{\sum_{k=0}^{m}{}^{*} p_{m,k}(x)\odot f_2\left(\frac{k+\alpha}{m+\beta}\right)}_{{}^{\mathcal{F}}L_m^{(\alpha,\beta)}(f_2;x)}.
\end{aligned}$$

The positivity: Let $f\preceq g$, where $f,g\in C_{\mathcal{F}}[0,1]$.
This implies

$$f\left(\frac{k+\alpha}{m+\beta}\right)\preceq g\left(\frac{k+\alpha}{m+\beta}\right),\ k=\overline{0,m},$$

and we deduce

$$\sum_{k=0}^{m}{}^{*} p_{m,k}(x)\odot f\left(\frac{k+\alpha}{m+\beta}\right)\preceq\sum_{k=0}^{m}{}^{*} p_{m,k}(x)\odot g\left(\frac{k+\alpha}{m+\beta}\right).$$

Consequently,

$${}^{\mathcal{F}}L_m^{(\alpha,\beta)}(f;x)\preceq{}^{\mathcal{F}}L_m^{(\alpha,\beta)}(g;x).$$

□

In order to give our main result we need the following theorem.

Theorem 2 ([5]) *Let $A=(a_{j,n})$ be a non-negative regular summability matrix and let $(L_n)_{n\in\mathbb{N}}$ be a sequence of fuzzy positive linear operators from $C_{\mathcal{F}}[a,b]$ into itself. Assume that there exists a corresponding sequence $(\widetilde{L_n})_{n\in\mathbb{N}}$ of positive linear operators from $C[a,b]$ into itself with the property* (2)*. Assume further that*

$$st_A-\lim_n \|\widetilde{L_n}(e_i)-e_i\| = 0 \quad \text{for each} \quad i=0,1,2.$$

Then, for all $f\in C_{\mathcal{F}}[a,b]$, we have

$$st_A-\lim_n D^*(L_n(f),f)=0. \tag{3}$$

Theorem 3 *If the sequence* $({}^{\mathcal{F}}L_m^{(\alpha,\beta)}f)_{m\in\mathbb{N}}$ *defined by* (1) *satisfies the conditions*

$$st_A - \lim_m \|\widetilde{L_m}^{(\alpha,\beta)}(e_i) - e_i\| = 0,\ \ i = 0,1,2, \tag{4}$$

then

$$st_A - \lim_m D^*({}^{\mathcal{F}}L_m^{(\alpha,\beta)}, f) = 0. \tag{5}$$

Proof. Since

$$(\widetilde{L_m}e_0)(x) = \sum_{k=0}^{m} \binom{m}{k} x^k(1-x)^{m-k} = 1, \tag{6}$$

clearly

$$st_A - \lim_m \|\widetilde{L_m}e_0 - e_0\| = 0. \tag{7}$$

We can also write

$$\begin{aligned}(\widetilde{L_m}e_1)(x) &= \sum_{k=0}^{m} \binom{m}{k} x^k(1-x)^{m-k}\frac{k}{m+\beta} + \sum_{k=0}^{m} \binom{m}{k} x^k(1-x)^{m-k}\frac{\alpha}{m+\beta}\\ &= x + \frac{\alpha-\beta x}{m+\beta}.\end{aligned} \tag{8}$$

Consequently, we get

$$\|\widetilde{L_m}e_1 - e_1\| \le \frac{\alpha}{m+\beta} + \frac{\beta}{m+\beta}.$$

For a given $\varepsilon > 0$, we consider the sets

$$D := \{m : \|\widetilde{L_m}e_1 - e_1\| \ge \varepsilon\},$$

$$D_1 := \{m : \frac{\alpha}{m+\beta} \ge \frac{\varepsilon}{2}\},\quad D_2 := \{m : \frac{\beta}{m+\beta} \ge \frac{\varepsilon}{2}\}.$$

It is obvious that $D \subset D_1 \cup D_2$. Consequently, for each $j \in \mathbb{N}$, we get

$$\sum_{m\in D} a_{j,m} \le \sum_{m\in D_1} a_{j,m} \le \sum_{m\in D_2} a_{j,m}. \tag{9}$$

Since $st_A - \lim_m \frac{\alpha}{m+\beta} = st_A - \lim_m \frac{\beta}{m+\beta} = 0$, taking in (9) the limit as j tends to infinity, we conclude

$$\lim_{j\to\infty} \sum_{m\in D} a_{j,m} = 0.$$

This identity implies

$$st_A - \lim_m \|\widetilde{L_m}e_1 - e_1\| = 0. \tag{10}$$

Further on, we obtain

$$\begin{aligned}\widetilde{L_m}(e_2, x) &= \sum_{k=0}^{m} \binom{m}{k} x^k(1-x)^{m-k}\frac{k^2}{(m+\beta)^2} + \sum_{k=0}^{m} \binom{m}{k} x^k(1-x)^{m-k}\frac{2k\alpha}{(m+\beta)^2} +\\ &+ \sum_{k=0}^{m} \binom{m}{k} x^k(1-x)^{m-k}\frac{\alpha^2}{(m+\beta)^2}\\ &= \frac{m^2}{(m+\beta)^2}\left(x^2 + \frac{x(1-x)}{m}\right) + \frac{2\alpha m}{(m+\beta)^2}x + \frac{\alpha^2}{(m+\beta)^2}\\ &= x^2 + \frac{xm - x^2m + 2\alpha xm + \alpha^2 - \beta^2x^2 - 2\beta x^2 m}{(m+\beta)^2}\\ &= x^2 + \frac{mx(1-x) + (\alpha-\beta x)(2mx + \alpha + \beta x)}{(m+\beta)^2},\end{aligned} \tag{11}$$

which implies

$$\|\widetilde{L_m}e_2 - e_2\| \le \frac{m}{4(m+\beta)^2} + \frac{(\alpha+\beta)(2m+\alpha+\beta)}{(m+\beta)^2}.$$

Reasoning in a similar manner as above, one obtains

$$st_A - \lim_m \|\widetilde{L_m}e_2 - e_2\| = 0. \tag{12}$$

On the basis of (7), (10), (12), taking into account Theorem 2, we arrive at (5). The proof is complete. □

4 Examples

In the following section we give an example of fuzzy positive and linear operators starting from fuzzy Bernstein-Stancu operators. These new operators satisfy a generalized fuzzy Korovkin theorem.

First of all, we recall a generalization of Theorem 1 using matrix summability method.

Theorem 4 ([4]) *Let $\mathcal{A} = (A^n)_{n\ge 1}$ be a sequence of infinite non-negative real matrices such that*

$$\sup_{n,k} \sum_{j=1}^{\infty} a_{kj}^n < \infty \tag{13}$$

and let $(L_j)_{j\in\mathbb{N}}$ be a sequence of fuzzy positive linear operators from $C_{\mathcal{F}}[a,b]$ into itself. Assume that there exists a corresponding sequence $(\widetilde{L_j})_{j\in\mathbb{N}}$ of positive linear operators from $C[a,b]$ into itself satisfying (2). *Assume further that*

$$\lim_{k\to\infty} \left\| \sum_{j=1}^{\infty} a_{kj}^n \widetilde{L_j}(e_i) - e_i \right\| = 0, \quad \text{for each} \quad i = 0,1,2, \tag{14}$$

uniformly in n. Then, for all $f \in C_{\mathcal{F}}[a,b]$, we have

$$\lim_{k\to\infty} D^* \left(\sum_{j=1}^{\infty} a_{kj}^n L_j(f), f \right) = 0,$$

uniformly in n.

Example 1 *Let $(u_j)_{j\in\mathbb{N}}$ a sequence almost convergent to zero such that $u_j \ge 0,\ \forall\ j \in \mathbb{N}$.*
For example, $u_j = 1 + (-1)^j$ is a special sequence which is almost convergent but is not statistically convergent.
By using our general sequence $(u_j)_{j\in\mathbb{N}}$ and the fuzzy Bernstein-Stancu operators we give an example of fuzzy positive and linear operators defined on $C_{\mathcal{F}}[0,1]$ which satisfy Theorem 4.
So, let

$$\mathcal{N}_j(f;x) = u_j \odot^{\mathcal{F}} L_m^{(\alpha,\beta)}(f;x),\ j\in\mathbb{N},\ x\in\mathbb{R},\ f\in C_{\mathcal{F}}[0,1]. \tag{15}$$

The corresponding real positive linear operators are given by

$$\widetilde{N_j}(f_\pm^{(r)};x) = u_j \sum_{k=0}^{j} \begin{pmatrix} j \\ k \end{pmatrix} x^k(1-x)^{j-k} \cdot f_\pm\left(\frac{k+\alpha}{j+\beta}\right),\quad j\in\mathbb{N},\ x\in\mathbb{R},\ f_\pm^{(r)}\in C[0,1]. \tag{16}$$

On the basis of (6), (8), (11), *we observe that*

$$\widetilde{N_j}(e_0,x) = u_j,$$

$$\widetilde{N_j}(e_1,x) = u_j\left[x+\frac{\alpha-\beta x}{m+\beta}\right],$$

$$\widetilde{N_j}(e_2,x) = u_j\left[x^2+\frac{mx(1-x)+(\alpha-\beta x)(2mx+\alpha+\beta x)}{(m+\beta)^2}\right].$$

Since $(u_j)_{j\in\mathbb{N}}$ *is almost convergent to zero, we get*

$$\lim_{k\to\infty}\|\sum_{j=1}^{\infty}a_{kj}^n\widetilde{N_j}(e_i)-e_i\|=0,\quad i=0,1,2,\quad \textit{uniformly in } n.$$

Consequently, from Theorem 4 *we have*

$$\forall f\in C_{\mathcal{F}}[0,1],\quad \lim_{k\to\infty}D^*(\sum_{j=1}^{\infty}a_{kj}^nN_j(f),f)=0,\quad \textit{uniformly in } n.$$

Remark 3 *According to* ([3; *Theorem* 29]), *for our operators defined by* (15) *we have*

$$\sup_{x\in[0,1]}D\left((\mathcal{N}_jf)(x),f(x)\right)\le\left[3+\frac{m^3+4m\alpha^2(m-\beta^2}{4(m-\beta^2)(m+\beta)^2}\right]\omega_2^{(\mathcal{F})}\left(f,\frac{1}{\sqrt{m}}\right)+\frac{2(\alpha+\beta)\sqrt{m}}{m+\beta}\omega_1^{(\mathcal{F})}\left(f,\frac{1}{\sqrt{m}}\right).$$

5 Conclusions and Future Work

In this paper we have proved that the fuzzy Bernstein -Stancu operators satisfy a fuzzy Korovkin theorem and we have presented an example of fuzzy positive and linear operators defined on $C_{\mathcal{F}}[0,1]$ which satisfy the fuzzy Korovkin theorem.

In the future we will explore the area of fuzzy positive and linear operators and we will introduce new classes of fuzzy positive and linear operators. Moreover, we will study the statistical convergence of sequences of these type of operators and we will extend these properties to multidimensional case.

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